$$\approx \quad e^{\ln a} + \frac{1-a}{2an} e^{\ln a} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

$$e^{\phi_n} \quad \approx \quad a + \frac{1-a}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

With this result it can now be seen that

$$a - e^{\phi_n} \approx \frac{a-1}{2n} - \mathcal{O}\left(\frac{1}{n^2}\right) \text{ and}$$
$$n\left(a - e^{\phi_n}\right) \approx \frac{a-1}{2} - \mathcal{O}\left(\frac{1}{n}\right).$$

Now the limit is easy to compute and is given by

$$\lim_{n\to\infty} n\left(a - e^{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{an}}\right) = \frac{a-1}{2}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Sagueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Vergata, Rome, Italy; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

5257: *Proposed by Pedro H.O. Pantoja, UFRN, Brazil*

Prove that:

$$1 + \frac{1}{2}\cdot\sqrt{1+\frac{1}{2}} + \frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}} + \cdots + \frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} \sim \ln(n),$$

where $f(x) \sim g(x)$ means $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$.

Solution 1 by Arkady Alt, San Jose, CA

Let $S_n = 1 + \frac{1}{2}\cdot\sqrt{h_2} + \frac{1}{3}\cdot\sqrt[3]{h_3} + ... + \frac{1}{n}\cdot\sqrt[n]{h_n}$,where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.

Since $\frac{1}{k+1} < \ln(k+1) - \ln k < \frac{1}{k}(\iff \left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1})$ then

$\sum_{k=1}^{n}(\ln(k+1) - \ln k) \; < \; h_n \quad \iff \quad \ln(n+1) \; < \; h_n$ and $h_k - 1 \; < \; \sum_{k=2}^{n}(\ln k - \ln(k-1)) \quad \iff$ $h_n < 1 + \ln n$

and, therefore, $\dfrac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = \dfrac{\frac{\sqrt[n]{h_n}}{n}}{\ln\left(1+\frac{1}{n-1}\right)} =$

$$\frac{\sqrt[n]{h_n}}{\ln\left(1+\frac{1}{n-1}\right)^n} \in \left(\frac{\sqrt[n]{\ln(n+1)}}{\ln\left(1+\frac{1}{n-1}\right)^n}, \frac{\sqrt[n]{\ln n+1}}{\ln\left(1+\frac{1}{n-1}\right)^n}\right).$$

Since $\lim_{n\to\infty} \sqrt[n]{\ln(n+1)} = 1,\ \lim_{n\to\infty} \sqrt[n]{1+\ln n} = 1, \lim_{n\to\infty} \ln\left(1+\frac{1}{n-1}\right)^n = 1$ then

$\lim_{n\to\infty} \frac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1$ and by Stolz Theorem we obtain

$$\lim_{n\to\infty} \frac{S_n}{\ln n} = \lim_{n\to\infty} \frac{S_n - S_{n-1}}{\ln n - \ln(n-1)} = 1.$$

Solution 2 by Ángel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain

Let L be the $\lim_{n\to\infty} \frac{1+\frac{1}{2}\cdot\sqrt{1+\frac{1}{2}}+\frac{1}{3}\cdot\sqrt[3]{1+\frac{1}{2}+\frac{1}{3}}+\cdots+\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)}$.

Since $\lim_{n\to\infty} \ln(n) = \infty$, by the Stolz-Cesàro theorem,

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln(n)-\ln(n-1)} \\ &= \lim_{n\to\infty} \frac{\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{\ln\left(\frac{n}{n-1}\right)^n}. \end{aligned}$$

Note that $\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} = 1$, by the Stolz-Cesàro theorem, and also that $\lim_{n\to\infty} \ln\left(\frac{n}{n-1}\right)^n = 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is well known that $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + O(1)$ as $n\to\infty$ and $\ln(1+x) = x + O\left(x^2\right),\ e^x = 1 = x + O\left(x^2\right)$ as $x\to 0$. Hence

$$\frac{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{n} = \frac{\ln\ln n}{n} + O\left(\frac{1}{n\ln n}\right)$$

and

$$\frac{1}{n}\cdot\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} = \frac{1}{n}e^{\ln\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)/n} = \frac{1}{n}\left(1+\frac{\ln\ln n}{n} + O\left(\frac{1}{n\ln n}\right)\right).$$